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DYNAMIC AND VIBROACOUSTIC METHODS

Statistical Characterization of Estimates of Relative Crack Dimensions Based on the Method of Higher Harmonics

N. I. Bouraou and A. N. Tyapchenko

National Technical University of Ukraine

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Abstract—Fatigue cracks are diagnosed and characterized using a vector of parameters composed of the ratios between spectral amplitudes of the higher and fundamental harmonics in inherent acoustic modes of an object. Using the method of maximal likelihood, we have obtained estimates of the relative crack dimensions and analyzed their statistical characteristics depending on the composition of the parameter vector and noise intensity in measurements.

The method of higher harmonics (MHH) [1] is used in the vibroacoustic diagnostic of fatigue flaws in blades of compressors and turbines of turbojet engines. The reason for this is the fact that inception and development of a fatigue cracks bring about changes in spectral characteristics of free or resonant mechanical oscillations in tested objects, in particular, an increase in spectral amplitudes of higher harmonics [2–4]. The diagnostic indicator of the crack presence in this method [3, 4] is a multidimensional vector whose components are the ratios between the amplitudes of higher harmonics and that of the fundamental harmonic in oscillations of the tested object (TO), and these vector components are the functions of the relative crack dimensions independent of the amplitude of the driving force. An increase in the relative crack dimension leads to higher magnitudes of the components of the diagnostic vector, which can be used in estimating nondestructively the crack characteristics on the base of amplitudes of the harmonics in the acoustic signal.

In the reported work, the estimates of the relative crack dimension were obtained by analyzing free oscillations of a blade with a surface fatigue crack, which were described mathematically using the analysis of bending oscillations of an elastic rod [5] with a fatigue flaw modeled as a discontinuity in the material leading to a difference between the compressive and tensile elastic moduli. In a general case, the relative change in the rod hardness on the half-periods of oscillations is a nonlinear integral function of the relative dimension and location of a crack, rod dimensions, and the presence of a concentrated stress. The theoretical analysis [5] indicated that the presence of a surface crack with a relative dimension of 0.05 to 0.3 in the strained zone brings about relative changes in the rod hardness with respect to bending oscillations in the lowest inherent mode within a range of 0.005–0.06, and in the case of an edge crack of the same dimensions, the relative change in the hardness ranges between 0.01 and 0.3.

In view of this, we have analyzed the problem of oscillations of an elastic rod whose nonlinearity is due to "breathing" of discontinuities in the material in terms of mechanical oscillations of a system with an asymmetrical characteristic of the elastic force. The relative crack dimension in this case can be derived from estimates of the relative change in the object hardness by performing a simple calculations with the help of previously known formulas [5]. Thus, the model of a TO is an oscillating system with one degree of freedom and a piece-wise linear characteristic of the elastic force without damping, and we describe its free oscillations by a Fourier series in terms of the higher harmonics of the fundamental frequency ω_0 [4]:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{K} a_k \cos k\omega_0 t\ , \tag{1}$$

where

$$\frac{a_0}{2} = A_*(1-\zeta)/\zeta\ ;\ A_* = 2\nu_0/\pi\omega_f;$$

$$a_k = \frac{2A\,(\zeta+1)^3(1-\zeta^2)}{\left[(\zeta+1)^2-4k^2\right]\left[(\zeta+1)^2-4\zeta^2k^2\right]\zeta}\cos\frac{\pi k}{\zeta+1};$$

$$\omega_0 = 2\zeta\omega_f/(\zeta+1);\ \zeta = (1-\vartheta)^{1/2};$$

ϑ is the relative difference between the hardness parameters on the half-periods of oscillations, ω_f is the frequency of free oscillations of the flawless object, and ν_0 is the frequency of the driving force.

As follows from these formulas, the frequency ω_0, the constant term a_0, and expansion factors a_k are functions of the relative change in the hardness, and the spectral amplitudes of the higher harmonics ($k > 1$) increase with ϑ [4].

The suggested diagnostic method is based on a multidimensional vector whose components are the ratios between the amplitudes of higher ($k = \overline{2, K}$) and fundamental ($k = 1$) harmonics

$$\mathbf{Y} = \left\{\frac{a_2}{a_1}, \frac{a_3}{a_1}, \ldots, \frac{a_k}{a_1}\right\}. \tag{2}$$

In accordance with the real conditions of measurements, we treat the spectral amplitudes of the fundamental harmonic and those labeled by index r ($r = \overline{2, K}$) obtained in the ith measurement, as random values ξ_{ik} characterized by the normal probability distribution:

$$P(\xi_{ik}) = \frac{1}{\sigma_k\sqrt{2\pi}}\exp\left[-\frac{(\xi_{ik}-m_k)^2}{2\sigma_k^2}\right]. \tag{3}$$

The mathematical expectations m_k of the small relative parameters $\vartheta = 0.01$–0.1 are approximated by polynomials of the first order with respect to ϑ:

$$m_1 = \mu_{01} + \mu_{11}\vartheta \text{ and } m_r = \mu_{0r} + \mu_{1r}\vartheta, \tag{4}$$

and the dispersion of the amplitude is estimated by the formula

$$\sigma_k^2 = N_0/\tau_p,$$

where N_0 is the white-noise intensity and τ_p is the Gaussian pulse width.

Let us estimate ϑ using the method of maximal likelihood [6]. The probability density $w(\eta)$ for the ratio $\eta_{ir} = \xi_{ir}/\xi_{i1}$ between two independent variables ξ_r and ξ_1 is determined by the formula [7]

$$w(\eta) = \int_{-\infty}^{+\infty} P_{\xi 1}(x_1)\,P_{\xi_r}(\eta x_1)\,|x_1|\,dx_1,\ \eta = \xi_r/\xi_1.$$

Taking into account equation (3), we obtain the following expression for $w(\eta_{ir}, \vartheta)$ in the ith measurement:

$$w(\eta_{ir}, \vartheta) = \frac{f_1(\eta_{ir}, \vartheta)}{\sqrt{2\pi}\left[f_2(\eta_{ir}, \vartheta)\right]^{3/2}}\exp\left\{\frac{\left[f_1(\eta_{ir}, \vartheta)\right]^2 - f_2(\eta_{ir}, \vartheta)\,f_3(\eta_{ir}, \vartheta)}{2\sigma_1^2\sigma_r^2 f_3(\eta_{ir}, \vartheta)}\right\}, \tag{5}$$

where

$$f_1(\eta_{ir}, \vartheta) = m_1\sigma_r^2 + m_r\sigma_1^2\eta_{ir},$$
$$f_2(\eta_{ir}, \vartheta) = \sigma_r^2 + \sigma_1^2\eta_{ir}^2,$$
$$f_3(\eta_{ir}, \vartheta) = m_1^2\sigma_r^2 + m_r^2\sigma_1^2.$$

For a sampling with independent elements, the equation of maximal likelihood has the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial \vartheta} \ln w(\eta_{ir}, \vartheta) = 0,$$

or, after transforming equation (5) with due account of (4), we obtain

$$\sum_{i=1}^{n} \frac{\vartheta^2 \sum_{j=0}^{3} A_{jr}\eta_{ir}^j + \vartheta \sum_{j=0}^{3} B_{jr}\eta_{ir}^j + \sum_{j=0}^{3} C_{jr}\eta_{ir}^j}{\vartheta \sum_{j=0}^{3} D_{jr}\eta_{ir}^j + \sum_{j=0}^{3} F_{jr}\eta_{ir}^j} = 0, \quad (6)$$

where

$$A_{3r} = -\sigma_1{}^2\mu^2{}_{11}\mu_{1r};\ A_{2r} = 2\sigma_1{}^2\mu_{11}\mu^2{}_{1r} - \sigma_r{}^2\mu_{11}{}^3;$$

$$A_{1r} = 2\sigma_r{}^2\mu_{1r}\mu^2{}_{11} - \sigma_1{}^2\mu^3{}_{1r};\ A_{0r} = -\sigma_r{}^2\mu_{11}\mu^2{}_{1r};$$

$$B_{3r} = -\sigma_1{}^2\mu_{11}b;\ B_{2r} = -2\sigma_r{}^2\mu_{01}\mu^2{}_{11} + 2\sigma_1{}^2\mu_{0r}\mu_{1r}\mu_{11} + \sigma_1{}^2\mu_{1r}\cdot b;$$

$$B_{1r} = 2\sigma_r{}^2\mu_{01}\mu_{11}\mu_{1r} - 2\sigma_1{}^2\mu_{0r}\mu^2{}_{1r} + \sigma_r\mu_{11}\cdot b;\ B_{0r} = -\sigma_r{}^2\mu_{1r}\cdot b;$$

$$C_{3r} = \sigma_1{}^4\mu_{1r} - \sigma_1{}^2\mu_{11}\mu_{01}\mu_{0r};\ C_{2r} = \sigma_1{}^2\sigma_r{}^2\mu_{11} - \sigma_r{}^2\mu_{11}\mu_{01}{}^2 + \sigma_1{}^2\mu_{0r}\cdot b;$$

$$C_{1r} = \sigma_1{}^2\sigma_r{}^2\mu_{1r} - \sigma_1{}^2\mu_{1r}\mu_{0r}{}^2 + \sigma_r{}^2\mu_{01}\cdot b;\ C_{0r} = \sigma_r{}^4\mu_{11} - \sigma_r{}^2\mu_{1r}\mu_{01}\mu_{0r};$$

$$D_{3r} = \sigma_1{}^4\mu_{1r};\ D_{2r} = \sigma_1{}^2\sigma_r{}^2\mu_{11};\ D_{1r} = \sigma_1{}^2\sigma_r{}^2\mu_{1r};\ D_{0r} = \sigma_r{}^4\mu_{11};$$

$$F_{3r} = \sigma_1{}^4\mu_{0r};\ F_{2r} = \sigma_1{}^2\sigma_r{}^2\mu_{01};\ F_{1r} = \sigma_1{}^2\sigma_r{}^2\mu_{0r};\ F_{0r} = \sigma_r{}^4\mu_{01};$$

$$b = \mu_{01}\mu_{1r} + \mu_{01}\mu_{11}.$$

Thus, the resulting equation of maximal likelihood in the general form (6) is the nth sum of ratios between the polynomials of the second and first orders in the sought-for estimate of ϑ, where n is the number of measurements.

In the case of one measurement ($n = 1$), the solution of equation (6) with respect to $\hat{\vartheta}$ for each component of vector (2) was obtained analytically as a solution of a quadratic equation [8], and at $n > 1$ equation (6) was solved numerically, and the calculation was performed, in particular, for $n = 2$, taking into account the limited accuracy of two measurements in the form

$$\eta_{1r} = \frac{m_r + \sigma^2_{\text{meas}}}{m_1 - \sigma^2_{\text{meas}}};\ \eta_{1r} = \frac{m_r - \sigma^2_{\text{meas}}}{m_1 + \sigma^2_{\text{meas}}},$$

where $\sigma^2_{\text{meas}} = 10^{-7}$.

The calculations of the dispersion in the relative hardness difference $\vartheta = 0.05$ on the base of each component ($K = 10$) of the higher-harmonic vector (2) at different measurement dispersions due to noise for $n = 1$ are given in table. At $n = 2$ the results are very close to the data in table (the dispersion in the second, fourth, and sixth harmonics reduces by 1 to 1.5% in the range of σ_i^2 under consideration), and the regularities of their changes with both the components of the characteristic vector and noise intensity are basically the same.

As is shown by the data given in table, the dispersions of vector components corresponding to the odd higher harmonics are considerably higher than those corresponding to the even harmonics over the entire range of noise intensity in measurements. The dispersion of measurements also increases with the number k of the higher harmonic in expansion (1), which is caused by the lower signal-to-noise ratio at higher k with the constant dispersion σ_i^2. It follows from the analysis of solutions of the inverse problem that the components corresponding to the higher odd harmonics can be excluded because they yield less information than the higher even harmonics, which is in agreement with the results of the analysis of deformation cycles in an elastic rod containing a fatigue crack [5].

Table

a_i/a_1 \ σ_i^2	10^{-10}	10^{-9}	10^{-8}	10^{-7}	10^{-6}
2	3.35×10^{-8}	3.34×10^{-8}	3.32×10^{-8}	3.07×10^{-8}	1.09×10^{-8}
3	1.92×10^{-5}	1.27×10^{-5}	2.12×10^{-5}	7.13×10^{-3}	5.72×10^{-1}
4	3.18×10^{-8}	1.8×10^{-8}	9.62×10^{-8}	2.25×10^{-5}	2.36×10^{-3}
5	3.27×10^{-6}	4.82×10^{-4}	6.02×10^{-2}	3.27	7.06×10^{1}
6	6.53×10^{-10}	1.7×10^{-6}	2.11×10^{-4}	2.02×10^{-2}	1.30
7	2.93×10^{-4}	4.08×10^{-2}	2.40	5.55×10^{1}	7.39×10^{2}
8	1.93×10^{-6}	2.38×10^{-4}	2.27×10^{-2}	1.43	3.72×10^{1}
9	8.92×10^{-3}	6.91×10^{-1}	2.14×10^{1}	3.25×10^{2}	3.71×10^{3}
10	9.11×10^{-5}	8.96×10^{-3}	6.47×10^{-1}	2.03×10^{1}	3.10×10^{2}

In accordance with this conclusion, further analysis of the estimates of the changes in the TO hardness was conducted taking into account only $n = 1$ and the vector of amplitude ratios only for the higher even harmonics:

$$\mathbf{Y} = \left\{\frac{a_2}{a_1}; \frac{a_4}{a_1}; \ldots; \frac{a_{10}}{a_1}\right\}. \tag{7}$$

By deriving the relative change in the hardness from each component of vector (7), let us define a vector of estimates of ϑ in the form

$$\widehat{\Theta} = \left\{\hat{\vartheta}_2; \hat{\vartheta}_4; \ldots; \hat{\vartheta}_{10}\right\},$$

which allows us to estimate the mathematical expectation $m_{\hat{\Theta}}$ and dispersion $D_{\hat{\Theta}}$ of the estimates of the relative changes in hardness for a given measurement.

Figure 1 plots $\log D_{\hat{\Theta}}$ versus the noise intensity in measurements of the relative change in hardness $\vartheta = 0.05$ for the following three characteristic vectors defined in accordance with equation (7):

$$\mathbf{Y}_1 = \left\{\frac{a_2}{a_1}; \frac{a_4}{a_1}\right\}; \mathbf{Y}_2 = \left\{\frac{a_2}{a_1}; \frac{a_4}{a_1}; \frac{a_6}{a_1}\right\}; \mathbf{Y}_3 = \left\{\frac{a_2}{a_1}; \frac{a_4}{a_1}; \frac{a_6}{a_1}; \frac{a_8}{a_1}\right\}.$$

One can see that the estimate based on vector $\mathbf{Y}_1$ has the lowest dispersion, whereas the accuracy of estimates based on all the three vectors degrades as the noise intensity in measurements increases.

Since an increase in the relative hardness change ϑ for a fixed intensity of the noise in measurements is equivalent to an increase in the signal-to-noise ratio, it is advisable to use in the statistical analysis of estimates of $\Theta = f(\vartheta, \sigma_i^2)$ the statistical parameter Q of estimate quality, which is widely used in digital methods of signal processing [9] and is defined as a ratio between the estimate dispersion and the mathematical expectation of this estimate squared:

$$Q = \frac{D_{\hat{\Theta}}}{\left[m_{\hat{\Theta}}\right]^2}.$$

The parameter Q is the inverted signal-to-noise ratio and directly related to the statistical stability of the estimate, and values $Q \ll 1$ correspond to smooth estimates with small dispersion. Graphs plotting $Q_1 = 20 \log Q$ as a function of the hardness change ϑ and noise intensity in measurements for estimated vectors of characteristic ratios $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$ are shown in Fig. 2.

The reported results allow one to select a vector of diagnostic parameters for which the estimate quality Q is no higher than a prescribed threshold value, depending on the conditions of measurements.

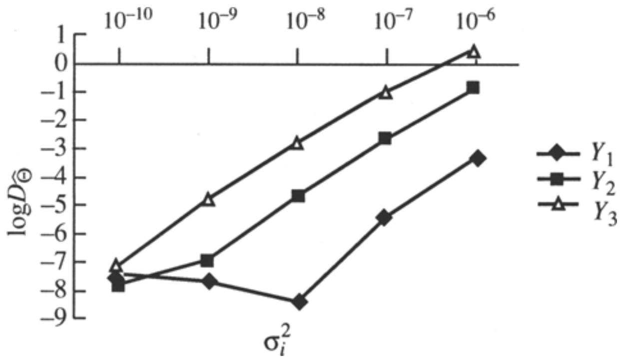

Fig. 1. Dispersion D_{Θ} versus noise intensity in measurements for the vectors of harmonic amplitude ratios Y_1, Y_2, and Y_3 ($\vartheta = 0.05$).

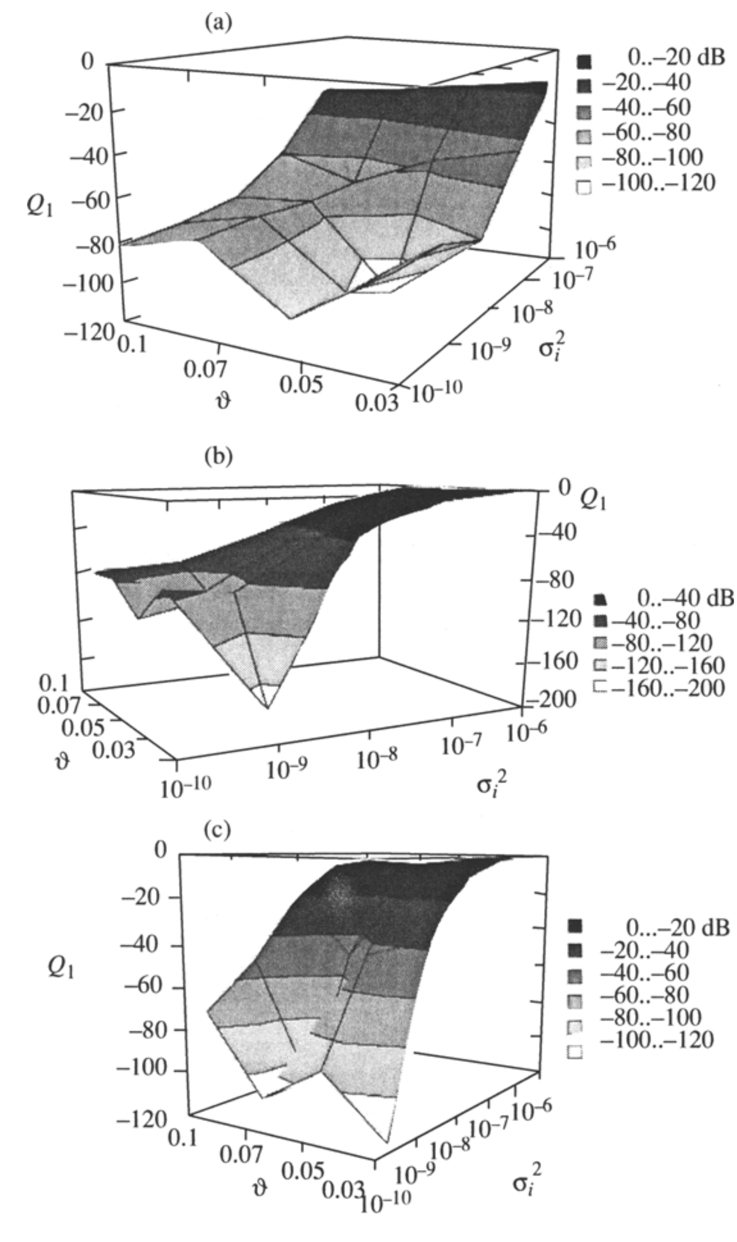

Fig. 2. Parameter Q_1 as a function of the measured parameter and dispersion of measurement noise amplitude estimated using vectors (a) Y_1, (b) Y_2, and (c) Y_3.

Thus, we have obtained, as a result of our investigation, an analytical expression and have analyzed the solution of the maximal likelihood equation with a view to estimating the relative change in the hardness of a tested object due to the presence of a fatigue crack. The analysis of statistical characteristics has enabled us to select vectors of ratios between higher and fundamental harmonics in free oscillations of the tested object which yield more accurate estimates of hardness. Since the signal-to-noise ratio for different components of the vectors is not constant in the range of relative hardness changes between 0.03 and 0.1, we have suggested to use the statistical characteristic of estimate quality for analyzing the accuracy, and its calculations for each selected vector of ratios have been given.

The reported results are original and, in combination with [5], have enabled us to come to the conclusion that efficient nondestructive testing of blades of gas-turbine jet engines with a view to detecting fatigue cracks in them is feasible.

REFRENCES

1. Zatsepin, N.N., *Metod vysshikh garmonik v nerazrushayushchem kontrole* (Method of Higher Harmonics in Nondestructive Testing), Minsk: Nauka i Tekhnika, 1980.
2. Karasev, V.A. and Roitman, A.B., *Dovodka ekspluatatsionnykh mashin. Vibroakustichesie metody* (Finishing up Operational Mechanisms. Vibro-Acoustic Methods), Moscow: Mashinostroenie, 1986.
3. Bouraou, N.I., Gelman, L.M., and Bertman, O.A., Low-frequency Vibroacoustic Free Oscillations Method for Nondestructive Testing and Evaluation of Fatigue Cracks, in *Proc. of the 2nd International Conf. on Computer Methods and Inverse Problems in Nondestructive Testing and Diagnostics*, Minsk, 20–23 October, 1998, DGZfP, Berlin, 1998.
4. Bouraou, N.I., Gelman, L.M., and Marchuk, P.I., Passive-Active Method of Vibroacoustic Diagnostic of Rotating Components in Aircraft Engines, in *Aviatsionno-kosmicheskaya tekhnika i tekhnologiya* (Aerospace Engineering and Technologies), Collection of Reseach Papers, issue 5, Kharkov State Aerospace University, 1998, pp. 374–378.
5. Matveev, V.V. and Bovsunovskii, A.P., Analysis of Efficient Methods of Spectral Vibrational Diagnostics of Fatigue Damage to Structural Components. Report 2. Flextural Oscillations. An Analytical Solution, *Probl. Prochnosti*, 1998, no. 6, pp. 9–22.
6. Levin, B.Z., *Teoreticheskie osnovy radiotekhniki* (Theoretical Principles of Radio-Electronics), in three volumes, vol. 2, 2nd edition, Moscow: Sovetskoe Radio, 1975.
7. Tikhonov, V.I., *Statisticheskaya radiotekhnika* (Statistical Radio-Electronics), Moscow: Sovetskoe Radio, 1966.
8. Bouraou, N.I., Tyapchenko, A.N., and Gelman, L.M., Assessment of Cracks in Blades of Gas-Turbine Engines Based on Spectral Amplitudes of Acoustic Signals, in *Aviatsionno-kosmicheskaya tekhnika i tekhnologiya* (Aerospace Engineering and Technologies), Collection of Reseach Papers, issue 9, Kharkov State Aerospace University, 1998, pp. 284–288.
9. Marple, S.L., Jr., *Digital Spectral Analysis with Applications*, Englewood Cliffs, New Jersey 07632: Prentice-Hall, Inc., 1987. Translated from English into Russian under the title *Tsifrovoi spektral'nyi analiz i ego prilozheniya*, Moscow: Mir, 1990.